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Standard Young tableaux and character generators of classical Lie groups

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Abstract. Character generators are derived for each of the classical groups SU(k), SO($2k+1$), Sp($2k$) and SO($2k$). In order to do this, use is made of various generalisations of standard Young tableaux. One set of these is in one-to-one correspondence with the contributions to the characters of each irreducible representation and the other, involving shifted Young diagrams, provides a means of enumerating those maximal chains in the fundamental posets of each group which serve to generate, via their descent subchains, all possible multichains. The character generators are then written down in terms of a suitable labelling of the poset elements. Illustrative examples are given.

1. Introduction

It is well known that the character of the irreducible representation λ_G of the compact semi-simple Lie group G having highest weight $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k)$ is given by the formula due to Weyl (1926):

$$\chi^{\lambda_G}(\boldsymbol{\phi})=\left(\sum_{S\in W_G}(\det S)\exp \mathrm{i}S(\boldsymbol{\lambda}+\boldsymbol{\delta})\cdot\boldsymbol{\phi}\right)\Big/\left(\sum_{S\in W_G}(\det S)\exp \mathrm{i}S\boldsymbol{\delta}\cdot\boldsymbol{\phi}\right), \tag{1.1}$$

where $\boldsymbol{\phi}=(\phi_1, \phi_2, \ldots, \phi_k)$ specifies the conjugacy classes of G, $\boldsymbol{\delta}$ is half the sum of the positive roots of the Lie algebra corresponding to G and the summation is carried out over all the elements S of the Weyl group W_G of G.

The character generator for such a group G is defined by

$$X_G(\boldsymbol{c}, \boldsymbol{\phi})=\sum_{\lambda_G}\chi^{\lambda_G}(\boldsymbol{\phi})\boldsymbol{\alpha}^{\boldsymbol{\lambda}} \tag{1.2}$$

where

$$\boldsymbol{\alpha}^{\boldsymbol{\lambda}}=\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\ldots\alpha_k^{\lambda_k}. \tag{1.3}$$

Interest in such generating functions has recently been stimulated by Patera and Sharp (1979) who showed that they could be used, not only to express characters of irreducible representations in the form

$$\chi^{\lambda_G}(\boldsymbol{\phi})=\sum_{\boldsymbol{w}}M_{\boldsymbol{w}}^{\lambda_G}\exp \mathrm{i}(\boldsymbol{w}\cdot\boldsymbol{\phi}), \tag{1.4}$$

where $\boldsymbol{w}=(w_1, w_2, \ldots, w_k)$ is a weight vector of G and $M_{\boldsymbol{w}}^{\lambda_G}$ is the multiplicity of

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this weight in the irreducible representation λ_G, but also to determine further generating functions such as those associated with group–subgroup branching rules and the decomposition of tensor products.

Although the generating function (1.2) for characters may be constructed using Weyl's character formula (1.1) this leads to considerable overcounting with mutual cancellations of unwanted terms. It is desirable to express the character generator (1.2) as a sum of positive terms.

In the case of the group SU(k) this was done by Stanley (1980) using a method which has as its key element the description of characters of SU(k) in terms of standard Young tableaux:

$$\chi^{\{\lambda\}}(\boldsymbol{\phi}) = \sum_{T^{\{\lambda\}}} \exp(\mathrm{i}\boldsymbol{w}\cdot\boldsymbol{\phi}), \tag{1.5}$$

where the summation is carried out over all standard Young tableaux $T^{\{\lambda\}}$ and $\boldsymbol{w}$ is the weight vector corresponding to $T^{\{\lambda\}}$.

In this paper it is this method which is extended to cover the remaining classical groups SO($2k+1$), Sp($2k$) and SO($2k$). Preliminary statements of the results for Sp($2k$) (King 1981) and for SO($2k$) and SO($2k+1$) (King and El-Sharkaway 1982) have been given elsewhere. In order to establish the validity of the results it is necessary to give a complete description of the characters of irreducible representations of these groups in terms of various types of standard Young tableaux. This has recently been done (King and El-Sharkaway 1983). The crucial result takes the form

$$\chi^{\lambda_G}(\boldsymbol{\phi}) = \sum_{T^{\lambda_G}} 2^{\beta_G} \exp(\mathrm{i}\boldsymbol{w}\cdot\boldsymbol{\phi}) \tag{1.6}$$

where the summation is carried out over all the standard Young tableaux T^{λ_G}, $\boldsymbol{w}$ is once more the weight vector of T^{λ_G} and β_G is a duplication parameter required only when dealing with the groups SO($2k+1$) and SO($2k$).

Unfortunately β_G depends not only upon G but also upon T^{λ_G} so that for the purpose of deriving the character generator (1.2) it is convenient to introduce augmented standard Young tableaux $T_\varepsilon^{\lambda_G}$ which, it will be shown, enable (1.6) to be replaced by

$$\chi^{\lambda_G}(\boldsymbol{\phi}) = \sum_{T_\varepsilon^{\lambda_G}} \exp(\mathrm{i}\boldsymbol{w}\cdot\boldsymbol{\phi}). \tag{1.7}$$

This result is derived from (1.6) in § 2, which includes the precise definition of the augmented standard Young tableaux $T_\varepsilon^{\lambda_G}$ of each irreducible representation λ_G of each classical group G.

As stressed by Baclawski (1983) the standard Young tableaux associated with certain elementary representations of SU(k) form the elements of a finite partially ordered set (poset). The importance of this poset is that its multichains define all possible standard Young tableaux whilst its maximal chains provide a means of generating this complete set. Baclawski also described the corresponding poset for Sp($2k$). Here, in § 3, these posets and similar posets for both SO($2k$) and SO($2k+1$) are specified.

In § 4 these posets, associated with the column structure of the augmented standard Young tableaux of each of the classical groups, are then depicted diagrammatically, in low rank cases, in a Cartesian framework. The purpose of this is to illustrate that the edges of the diagram may be labelled so as to provide a geometric interpretation of the shifted Young tableaux introduced by Stanley (1980) in the case of SU(k). A similar geometric interpretation is given of the shifted Young tableaux (King 1981, King and El-Sharkaway 1982) appropriate to the other classical groups. In each case the shifted Young tableaux serve to specify the maximal chains of the corresponding poset from which all possible multichains may be generated.

These are then used in § 5 to derive the character generators (1.2) for each of the classical groups. The results are discussed in § 6 which includes some illustrative examples in which the character generator is expressed as succinctly as possible.

2. Augmented standard Young tableaux

The inequivalent irreducible representations λ_G of the classical groups G may be labelled by

$\{\lambda\}$ with $p \leqslant k-1$	for SU(k)
$[\lambda]$ and $[\Delta; \lambda]$ with $p \leqslant k$	for SO($2k+1$)
$\langle\lambda\rangle$ with $p \leqslant k$	for Sp($2k$)
$[\lambda]$ with $p \leqslant k-1$, $[\lambda]_\pm$ with $p=k$ and $[\Delta; \lambda]_\pm$ with $p \leqslant k$	for SO($2k$)

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0)$ has k components, the first p of which are integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$.

The highest weights $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k)$ of these irreducible representations λ_G are such that

for $\{\lambda\}, [\lambda]$ and $\langle\lambda\rangle$

$$\boldsymbol{\lambda}_i = \lambda_i \qquad \text{for } 1 \leqslant i \leqslant k,$$

for $[\lambda]_\pm$

$$\boldsymbol{\lambda}_i = \begin{cases} \lambda_i & \text{for } 1 \leqslant i \leqslant k-1 \\ \pm\lambda_k & \text{for } i = k, \end{cases}$$

for $[\Delta; \lambda]$

$$\boldsymbol{\lambda}_i = \lambda_i + \tfrac{1}{2} \qquad \text{for } 1 \leqslant i \leqslant k,$$

and for $[\Delta; \lambda]_\pm$

$$\boldsymbol{\lambda}_i = \begin{cases} \lambda_i + \frac{1}{2} & \text{for } 1 \leqslant i \leqslant k-1 \\ \pm(\lambda_i + \frac{1}{2}) & \text{for } i = k. \end{cases}$$

This notation is used elsewhere (King and El-Sharkaway 1983) to motivate the construction of various Young diagrams. Just as λ specifies a Young diagram F^λ consisting of p rows of boxes, of lengths $\lambda_1, \lambda_2, \ldots, \lambda_p$ left adjusted to a vertical line, so $\Delta; \lambda$ specifies a Young diagram $F^{\Delta;\lambda}$ formed by adjoining a column of half boxes of length k to the left of F^λ.

Furthermore (King and El-Sharkaway 1983), standard Young tableaux T^λ and $T^{\Delta;\lambda}$ may be formed by inserting in the boxes and half-boxes of the diagrams F^λ and $F^{\Delta;\lambda}$ various entries taken from some totally ordered set, S, subject to certain rules which depend upon the group under consideration. In the case of the groups SU(k), SO($2k+1$), Sp($2k$) and SO($2k$) the requisite sets, S, are:

$$S_A = \{1, 2, \ldots, k\} \qquad S_B = \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{k}, k, 0\} \qquad (2.1a, b)$$

$$S_C = \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{k}, k\} \qquad S_D = \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{k}, k\}. \qquad (2.1c, d)$$

The total ordering of the elements in these sets is defined by

$$\bar{1}<1<\bar{2}<2\ldots<\bar{k}<k<0. \tag{2.2}$$

If the number taken from the appropriate set S and entered in the box or half-box of the ith row and jth column of F^λ or $F^{\Delta;\lambda}$ is $\eta(i,j)$ then in all cases

$$\eta(i,j)\geqslant\eta(i,j-1) \qquad \text{for } i\geqslant 1 \text{ and } j\geqslant 2 \tag{2.3}$$

$$\eta(i,j)>\eta(i-1,j) \qquad \text{for } i\geqslant 2 \text{ and } j\geqslant 1. \tag{2.4}$$

These rules suffice for the entries in the standard Young tableaux of SU(k), but additional rules are required for the other groups. In particular for SO($2k+1$), Sp($2k$) and SO($2k$)

$$\eta(i,1)\geqslant\bar{i} \qquad \text{for } T^\lambda, \tag{2.5}$$

whilst for SO($2k+1$) and SO($2k$)

$$\eta(i,1)=\bar{i} \text{ or } i \qquad \text{for } T^{\Delta;\lambda}, \tag{2.6}$$

and

$$\eta(i,1)=\bar{i} \text{ and } \eta(i,j)=i \text{ implies } \eta(i-1,j)=\bar{i} \text{ for } T^\lambda \text{ and } T^{\Delta;\lambda}. \tag{2.7}$$

It is convenient to generalise these standard Young tableaux T^λ and $T^{\Delta;\lambda}$, with entries $\eta(i,j)$, by introducing augmented standard Young tableaux T^λ_ε and $T^{\Delta;\lambda}_\varepsilon$, with entries $\eta(i,j)_{\varepsilon(i,j)}$, where the subscripts $\varepsilon(i,j)$ are taken from the set

$$S_\varepsilon=\{\bar{1},1,0\} \tag{2.8}$$

equipped with the partial ordering

$$\bar{1}<0 \text{ and } 1<0. \tag{2.9}$$

For both SU(k) and Sp($2k$)

$$\varepsilon(i,j)=0, \tag{2.10}$$

whilst for both SO($2k+1$) and SO($2k$)

$$\varepsilon(i,j)=\begin{cases}1 & \text{if } \eta(i,j)=i \text{ and } \eta(i-1,j)\neq\bar{i}\\ \bar{1} & \text{if } \eta(i,j)=\bar{i}\\ 1 \text{ or } \bar{1} & \text{if } \eta(i,j)=i \text{ and } \eta(i-1,j)=\bar{i}\\ 0 & \text{if } \eta(i,j)\neq\bar{i} \text{ or } i.\end{cases} \tag{2.11}$$

In augmenting the entries $\eta(i,j)$ with these subscript $\varepsilon(i,j)$ it is required that:

$$\varepsilon(i,j)\geqslant\varepsilon(i,j-1) \qquad \text{for } i\geqslant 1 \text{ and } j\geqslant 2. \tag{2.12}$$

Thus in any row the subscripts are either all 0, or a sequence of 1's followed by 0's or a sequence of $\bar{1}$'s followed by 0's.

In general these rules are such that each $T^{\Delta;\lambda}$ determines a unique $T^{\Delta;\lambda}_\varepsilon$. However, each T^λ may determine more than one T^λ_ε. For example if

$$T^{(\Delta;531^2)}=\begin{array}{l}\bar{1}/\bar{1}\ \ \bar{4}\ \ \bar{4}\\ 2/\bar{3}\ \ 4\ \ 5\\ \bar{3}/3\\ \bar{4}/\\ 5/\end{array} \qquad \text{then } T^{(\Delta;531^2)}_\varepsilon=\begin{array}{l}\bar{1}_{\bar{1}}/\bar{1}_{\bar{1}}\ \ \bar{4}_0\ \ \bar{4}_0\\ 2_1/\bar{3}_0\ \ 4_0\ \ 5_0\\ \bar{3}_{\bar{1}}/3_{\bar{1}}\\ \bar{4}_{\bar{1}}/\\ 5_1/\end{array}$$

but if

$$T^{(531^2)} = \begin{array}{lllll} \bar{1} & \bar{2} & 2 & 2 & 4 \\ 2 & 2 & 3 & & \\ \bar{4} & & & & \\ 4 & & & & \end{array} \qquad \text{then } T_\varepsilon^{(531^2)} = \begin{array}{lllll} \bar{1}_{\bar{1}} & \bar{2}_0 & 2_0 & 2_0 & 4_0 \\ 2_1 & 2_1 & 3_0 & & \\ \bar{4}_0 & & & & \\ 4_1 & & & & \end{array} \qquad \text{with } \sigma = -1$$

$$\text{or } T_\varepsilon^{(531^2)} = \begin{array}{lllll} \bar{1}_{\bar{1}} & \bar{2}_0 & 2_0 & 2_0 & 4_0 \\ 2_1 & 2_{\bar{1}} & 3_0 & & \\ \bar{4}_0 & & & & \\ 4_{\bar{1}}. & & & & \end{array} \qquad \text{with } \sigma = +1$$

Similarly if

$$T^{(1^5)} = \begin{array}{l} 1 \\ \bar{3} \\ 3 \\ \bar{5} \\ 5 \end{array} \qquad \text{then } T_\varepsilon^{(1^5)} = \begin{array}{l} 1_1 \\ \bar{3}_0 \\ 3_1 \\ \bar{5}_0 \\ 5_1 \end{array} \quad \text{or} \quad \begin{array}{l} 1_1 \\ \bar{3}_0 \\ 3_{\bar{1}} \\ \bar{5}_0 \\ 5_{\bar{1}} \end{array} \qquad \text{with } \sigma = +1$$

$$\text{or } T_\varepsilon^{(1^5)} = \begin{array}{l} 1_1 \\ \bar{3}_0 \\ 3_1 \\ \bar{5}_0 \\ 5_{\bar{1}} \end{array} \quad \text{or} \quad \begin{array}{l} 1_1 \\ \bar{3}_0 \\ 3_{\bar{1}} \\ \bar{5}_0 \\ 5_1 \end{array} \qquad \text{with } \sigma = -1$$

where a signature parameter σ has been introduced for later convenience. The signature takes value +1 or −1 according as the number of entries augmented by the subscript $\bar{1}$ in the first column of the tableau, whether T_ε^λ or $T_\varepsilon^{\Delta;\lambda}$, is even or odd respectively. Thus

$$\sigma = \prod_{i \geq 1} (-1)^{\delta_{\bar{1}, \varepsilon(i,1)}} \tag{2.13}$$

These duplications are precisely what is required to enable the character of each irreducible representation λ_G of the classical group G under consideration to be expressed in the form (1.7). Each of the augmented standard Young tableaux $T_\varepsilon^{\lambda_G}$ associated with λ_G contributes the single term $\exp(\mathrm{i}\boldsymbol{w} \cdot \boldsymbol{\phi})$ to $\chi^{\lambda_G}(\boldsymbol{\phi})$ where the weight vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)$ is determined by the entries $\eta(i, j)$ of $T_\varepsilon^{\lambda_G}$. In fact (King and El-Sharkaway 1983),

$$w_i = n_i - n_{\bar{i}} \qquad \text{for } i = 1, 2, \ldots, k \tag{2.14}$$

where

$$n_e = \begin{cases} \sum_{i \geq 1, j \geq 1} \delta_{e, \eta(i,j)} & \text{for } T_\varepsilon^\lambda \\ \sum_{i \geq 1} \frac{1}{2} \delta_{e, \eta(i,1)} + \sum_{i \geq 1, j \geq 2} \delta_{e, \eta(i,j)} & \text{for } T_\varepsilon^{\Delta;\lambda} \end{cases} \tag{2.15}$$

for each entry e taken from the appropriate set S. The augmented standard Young tableaux $T_\varepsilon^{\lambda_G}$ associated with each particular irreducible representation λ_G are

for $\{\lambda\}$, $[\lambda]$ and $\langle\lambda\rangle$

$$T_\varepsilon^{\lambda_G} = T_\varepsilon^{\lambda}, \tag{2.16}$$

for $[\lambda]_\pm$

$$T_\varepsilon^{\lambda_G} = T_\varepsilon^{\lambda} \qquad \text{with } \sigma = \pm 1, \tag{2.17}$$

for $[\Delta;\lambda]$

$$T_\varepsilon^{\lambda_G} = T_\varepsilon^{\Delta;\lambda}, \tag{2.18}$$

and for $[\Delta;\lambda]_\pm$

$$T_\varepsilon^{\lambda_G} = T_\varepsilon^{\Delta;\lambda} \qquad \text{with } \sigma = \pm 1. \tag{2.19}$$

3. Fundamental posets of the classical groups

The fact that standard Young tableaux may be used to express the characters of irreducible representations of $SU(k)$ in the form (1.5) was the basis of the determination by Stanley (1980) of the character generator of this group. His technique made use of the column structure of these tableaux. This idea was exploited by Baclawski (1983) in extending the results to the case of the group $Sp(2k)$. This extension was possible because the duplication parameter β_G, appearing in (1.6), vanishes if G is $Sp(2k)$. In generalising these results still further to the remaining classical groups $SO(2k)$ and $SO(2k+1)$ it is merely necessary to make use of the augmented standard Young tableaux introduced in the previous section.

Following the procedure of Baclawski the first step is to introduce the fundamental posets $\boldsymbol{A}(k-1)$, $\boldsymbol{B}(k)$, $\boldsymbol{C}(k)$ and $\boldsymbol{D}(k)$ of $SU(k)$, $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ respectively. The poset elements are the augmented standard Young tableaux associated with the following elementary irreducible representations of these groups:

$$\begin{array}{ll} \{1^p\} \text{ with } 1 \leqslant p \leqslant k-1 & \text{for } SU(k) \\ [1^p] \text{ with } 1 \leqslant p \leqslant k & \text{for } SO(2k+1) \\ \langle 1^p\rangle \text{ with } 1 \leqslant p \leqslant k & \text{for } Sp(2k) \\ [1^p] \text{ with } 1 \leqslant p \leqslant k-1 \text{ and } [1^k]_\pm & \text{for } SO(2k). \end{array} \tag{3.1}$$

For example the posets of the low rank groups are given by:

$$\begin{array}{ll} SU(2) & \boldsymbol{A}(1) = \{1, 2\} \\ SU(3) & \boldsymbol{A}(2) = \left\{\begin{matrix} 1, & 2, & 3, & 1, & 1, & 2 \\ & & & 2 & 3 & 3 \end{matrix}\right\} \\ SO(3) & \boldsymbol{B}(1) = \{\bar{1}_-, 1_+, 0\} \\ SO(5) & \boldsymbol{B}(2) = \left\{\begin{matrix} \bar{1}_-, & 1_+, & \bar{2}, & 2, & 0, & \bar{1}_-, & \bar{1}_-, & 1_+, & 1_+, & \bar{2}, & \bar{2}, & \bar{1}_-, & 1_+, & \bar{2}, & 2 \\ & & & & & \bar{2}_- & 2_+ & \bar{2}_- & 2_+ & 2_- & 2_+ & 0 & 0 & 0 & 0 \end{matrix}\right\} \\ Sp(2) & \boldsymbol{C}(1) = \{\bar{1}, 1\} \end{array} \tag{3.2}$$

Sp(4) $\quad \boldsymbol{C}(2)=\left\{\bar{1}, 1, \bar{2}, 2, \begin{array}{c}\bar{1}\\ \bar{2}\end{array}, \begin{array}{c}\bar{1}\\ 2\end{array}, \begin{array}{c}1\\ \bar{2}\end{array}, \begin{array}{c}1\\ 2\end{array}, \begin{array}{c}\bar{2}\\ 2\end{array}\right\}$

SO(2) $\quad \boldsymbol{D}(1)=\{\bar{1}_-, 1_+\}$

SO(4) $\quad \boldsymbol{D}(2)=\left\{\bar{1}_-, 1_+, \bar{2}, 2, \begin{array}{c}\bar{1}_-\\ \bar{2}_-\end{array}, \begin{array}{c}\bar{1}_-\\ 2_+\end{array}, \begin{array}{c}1\\ \bar{2}_-\end{array}, \begin{array}{c}1\\ 2_+\end{array}, \begin{array}{c}\bar{2}\\ 2_-\end{array}, \begin{array}{c}\bar{2}\\ 2_+\end{array}\right\}.$

In displaying these posets the subscripts $\varepsilon(i, j)$ of value $\bar{1}$ and 1 have, for typographical convenience, been denoted by $-$ and $+$ respectively, whilst those of value 0 have been omitted altogether.

The elements of the posets are in one-to-one correspondence with the contributions to the characters of the elementary irreducible representations. Hence the number of elements in each poset may be expressed in terms of the dimensions of these representations. Since these dimensions are given by:

$$d_n\{1^p\}=d_n[1^p]=\binom{n}{p} \tag{3.3a}$$

$$d_n\langle 1^p\rangle=d_n\{1^p\}-d_n\{1^{p-2}\} \tag{3.3b}$$

and

$$d_{2k}[1^k]_\pm=\tfrac{1}{2}d_{2k}[1^k]=\tfrac{1}{2}\binom{2k}{k}, \tag{3.3c}$$

it follows that

$$|\boldsymbol{A}(k-1)|=\sum_{p=1}^{k-1} d_k\{1^p\}=2^k-2 \tag{3.4}$$

$$|\boldsymbol{B}(k)|=\sum_{p=1}^{k} d_{2k+1}[1^p]=2^{2k}-1 \tag{3.5}$$

$$|\boldsymbol{C}(k)|=\sum_{p=1}^{k} d_{2k}\langle 1^p\rangle=\binom{2k+1}{k}-1 \tag{3.6}$$

and

$$|\boldsymbol{D}(k)|=\sum_{p=1}^{k} d_{2k}[1^p]=2^{2k-1}-1+\tfrac{1}{2}\binom{2k}{k}. \tag{3.7}$$

These results are a generalisation to SO$(2k+1)$ and SO$(2k)$ of those given earlier by Baclawski (1983) for SU(k) and Sp$(2k)$.

The partial ordering applying to each poset is such that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of the poset then $\boldsymbol{x}<\boldsymbol{y}$ if and only if $\boldsymbol{xy}$ is a two column augmented standard Young tableau formed by the juxtaposition of the single column augmented standard Young tableaux $\boldsymbol{x}$ and $\boldsymbol{y}$.

The graph of each poset may then be obtained by associating a vertex, labelled by $\boldsymbol{x}$, with each poset element $\boldsymbol{x}$, and associating an edge with each pair of poset elements $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x}<\boldsymbol{y}$ and such that there exists no poset element $\boldsymbol{z}$ for which $\boldsymbol{x}<\boldsymbol{z}<\boldsymbol{y}$.

In each case the graph may be realised very conveniently in a Cartesian framework with the poset element $\boldsymbol{x}$ placed at the point

$$\boldsymbol{r}(\boldsymbol{x})=\sum_{i=1}^{k} r_i(\boldsymbol{x})\boldsymbol{e}_i, \tag{3.8}$$

where $\boldsymbol{e}_i$ for $i=1,2,\ldots,k$ are a set of mutually orthogonal unit vectors and where $r_i(\boldsymbol{x})$ is determined for each of the groups under consideration by:

$$\mathrm{SU}(k) \qquad r_i(\boldsymbol{x}) = k+1-j \qquad \text{if } x_i = j \text{ with } j \geqslant i \tag{3.9}$$

$$\mathrm{SO}(2k+1) \qquad r_i(\boldsymbol{x}) = \begin{cases} 2k+2-2j & \text{if } x_i = j \text{ with } j > i \\ 2k+3-2j & \text{if } x_i = \bar{j} \text{ with } j > i \\ 2k+2-2i & \text{if } x_i = i_+ \\ 2k+2-2i+\delta & \text{if } x_i = i_- \text{ or } \bar{i}_- \\ 1 & \text{if } x_i = 0 \end{cases} \tag{3.10}$$

$$\mathrm{Sp}(2k) \qquad r_i(\boldsymbol{x}) = \begin{cases} 2k+1-2j & \text{if } x_i = j \text{ with } j \geqslant i \\ 2k+2-2j & \text{if } x_i = \bar{j} \text{ with } j \geqslant i \end{cases} \tag{3.11}$$

$$\mathrm{SO}(2k) \qquad r_i(\boldsymbol{x}) = \begin{cases} 2k+1-2j & \text{if } x_i = \bar{j} \text{ with } j > i \\ 2k+2-2j & \text{if } x_i = j \text{ with } j > i \\ 2k+1-2i & \text{if } x_i = i_+ \\ 2k+1-2i+\delta & \text{if } x_i = i_- \text{ or } \bar{i}_- \end{cases} \tag{3.12}$$

where

$$x_i = \eta(i, 1)_{\varepsilon(i,1)} \tag{3.13}$$

is the entry in the ith row of the single column augmented standard Young tableaux $\boldsymbol{x}$. If there is no such entry, that is to say $i > p$ where p is the length of the column constituting $\boldsymbol{x}$, then

$$r_i(\boldsymbol{x}) = 0 \qquad \text{for } i > p. \tag{3.14}$$

The parameter δ in (3.10) and (3.12) is a small positive number sufficient merely to separate otherwise coincident elements in the graphs of the corresponding posets.

The Cartesian realisations of the posets of the low rank groups are displayed in figures 1–4 with the subscripts $\varepsilon(i, j)$ omitted for convenience.

The manner in which the posets are defined makes it clear that there is a one-to-one correspondence between the multichains $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q)$ with $\boldsymbol{x}_1 \leqslant \boldsymbol{x}_2 \leqslant \ldots \leqslant \boldsymbol{x}_q$ and the augmented standard Young tableaux T_ε^λ with $\lambda_1 = q$. This comes about through the identification of T_ε^λ with the juxtaposition of its columns $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_q$. Generalising this idea slightly the multichains $(\boldsymbol{x}_1^{1/2}, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{q+1})$ with $\boldsymbol{x}_1 \leqslant \boldsymbol{x}_2 \leqslant \ldots \leqslant \boldsymbol{x}_{q+1}$, where $\boldsymbol{x}_1^{1/2}$ is an augmented standard Young tableau of type T_ε^Δ, are in one-to-one correspondence with the augmented standard Young tableaux $T_\varepsilon^{\Delta;\lambda}$ with $\lambda_1 = q$. This generalisation of the notion of multichain is necessitated by the fact that in selecting the elementary representations whose augmented standard Young tableaux defined the elements of the fundamental posets, the fundamental spin representations Δ and $\Delta_\pm$ of $\mathrm{SO}(2k+1)$ and $\mathrm{SO}(2k)$, respectively, were omitted. Allowing half-elements, $\boldsymbol{x}_1^{1/2}$, of posets to appear in multichains provides the missing augmented standard Young tableaux of these spin representations. This establishes the one-to-one correspondence between the multichains of the fundamental poset of each classical group G and the augmented standard Young tableaux $T_\varepsilon^{\lambda_G}$ for all irreducible representations λ_G of G.

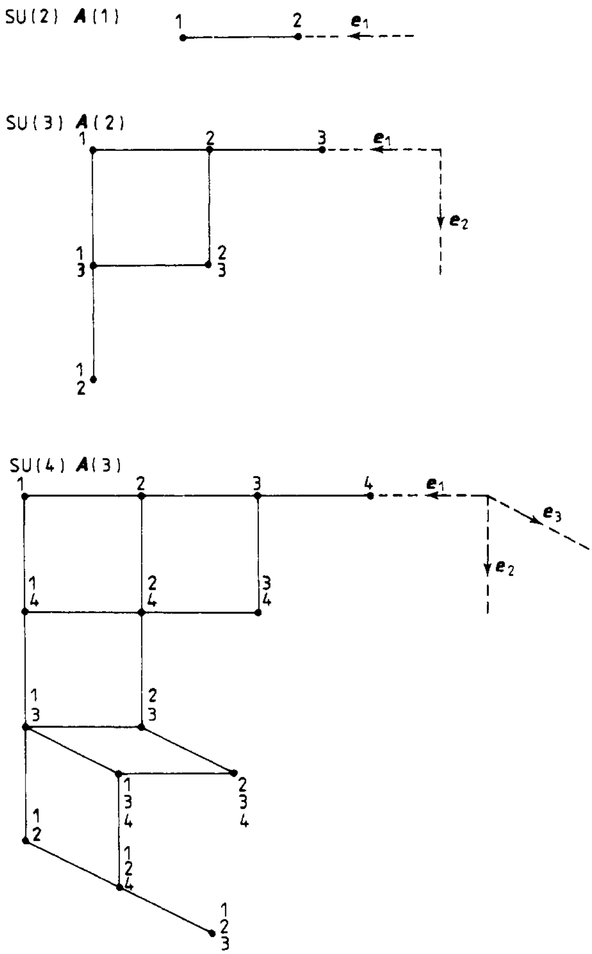


Figure 1. Fundamental posets $\boldsymbol{A}(k-1)$ of SU(k) for $k=2, 3, 4$.

4. Maximal chains and shifted Young tableaux

It is convenient to add to each poset $\boldsymbol{P}(k)$ a greatest element $\hat{\boldsymbol{x}}_1$ corresponding to an empty tableau, containing no boxes or half-boxes and therefore no entries. This element is such that:

$$\boldsymbol{x} < \hat{\boldsymbol{x}}_1 \qquad \text{for all } \boldsymbol{x} \in \boldsymbol{P}(k) \tag{4.1}$$

and in the graph of this poset the corresponding vertex is to be placed at the origin, so that

$$\boldsymbol{r}(\hat{\boldsymbol{x}}_1) = \boldsymbol{0}. \tag{4.2}$$

For each poset $\boldsymbol{P}(k)$ there exists a set of maximal chains $\boldsymbol{c} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$ with $\boldsymbol{x}_\alpha \in \boldsymbol{P}(k)$ for $\alpha = 1, 2, \ldots, m$ such that there is no element $\boldsymbol{z} \in \boldsymbol{P}(k)$ satisfying any one of the following conditions

$$\boldsymbol{z} < \boldsymbol{x}_1 \tag{4.3}$$

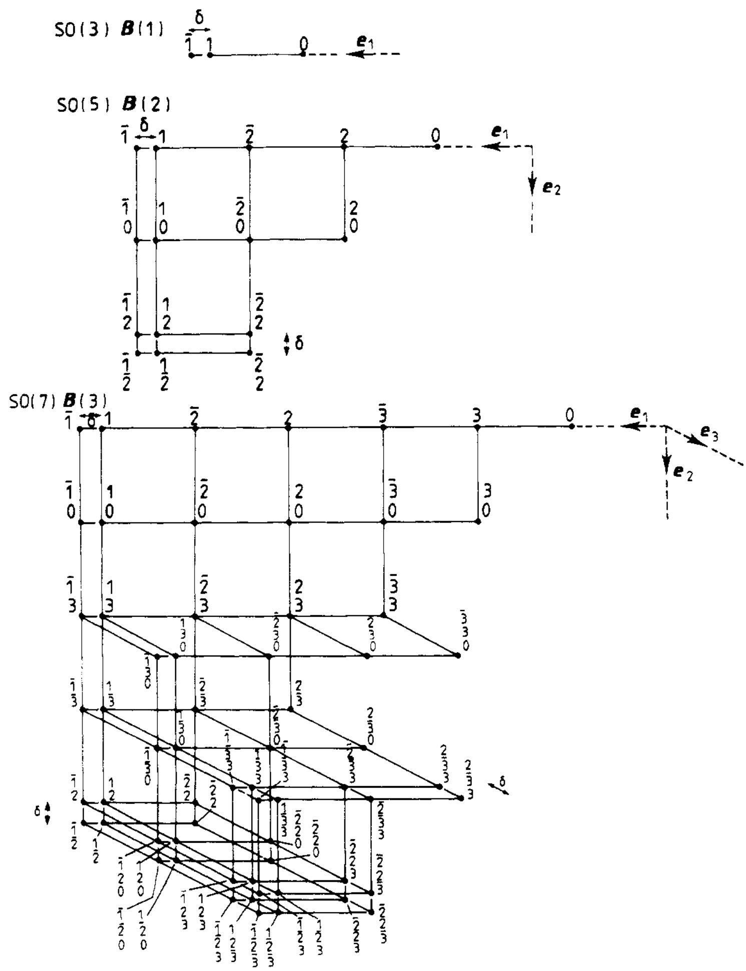


Figure 2. Fundamental posets $\boldsymbol{B}(k)$ of $SO(2k+1)$ for $k=1, 2, 3$.

or

$$\boldsymbol{x}_\alpha < \boldsymbol{z} < \boldsymbol{x}_{\alpha+1} \qquad \text{for } \alpha = 1, 2, \ldots, m \tag{4.4}$$

with

$$\boldsymbol{x}_{m+1} = \hat{\boldsymbol{x}}_1. \tag{4.5}$$

In passing along a maximal chain $\boldsymbol{c}$ from $\boldsymbol{x}_\alpha = \boldsymbol{x}$ to $\boldsymbol{x}_{\alpha+1} = \boldsymbol{y}$ for $\alpha = 1, 2, \ldots, m$ the constraint that there exists no $\boldsymbol{z}$ satisfying (4.4) is such that the edge joining $\boldsymbol{x}$ to $\boldsymbol{y}$ in the graph of the poset may be labelled unambiguously by

$$\lambda(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} i & \text{if } \boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{r}(\boldsymbol{y}) = \boldsymbol{e}_i \\ \bar{i} & \text{if } \boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{r}(\boldsymbol{y}) = (1+\delta)\boldsymbol{e}_i, \end{cases} \tag{4.6}$$

where use has been made of the notation of (3.9)–(3.12) and (4.2).

The labelling is thus a map from the edges of the graph to the set:

$$S = \{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{k}, k\}. \tag{4.7}$$

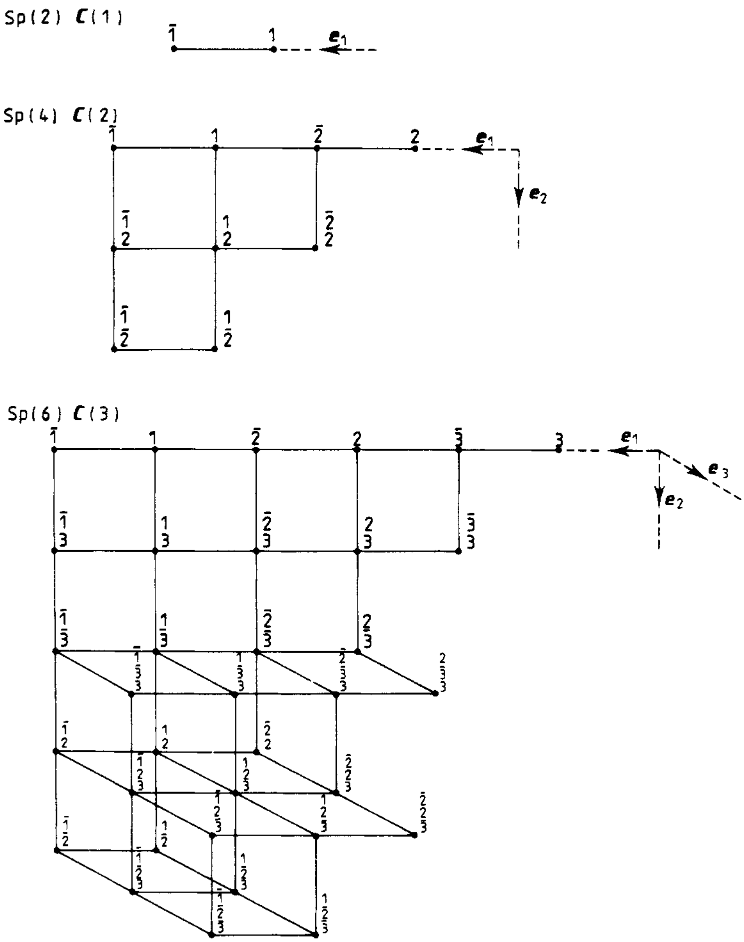


Figure 3. Fundamental posets $C(k)$ of Sp$(2k)$ for $k=1, 2, 3$.

The importance of such a labelling is that it enables a Yamanouchi symbol $Y(c)$ to be associated with each maximal chain $c=(x_1, x_2, \ldots, x_m)$ through the definition:

$$Y(c)=Y_1(c)\,Y_2(c)\ldots Y_m(c), \tag{4.8}$$

with

$$Y_\alpha(c)=\lambda(x_\alpha, x_{\alpha+1}) \qquad \text{for } \alpha=1,2,\ldots,m. \tag{4.9}$$

Since $r(x_{m+1})=r(\hat{x}_1)=\mathbf{0}$ it follows from (4.6) and (4.7) that if the sequence

$$Y_\alpha(c)\,Y_{\alpha+1}(c)\ldots Y_m(c) \tag{4.10}$$

contains a total of $y_i(x_\alpha)$ entries i and $y_{\bar{i}}(x_\alpha)$ entries $\bar{i}$ then

$$r_i(x_\alpha)=y_i(x_\alpha)+y_{\bar{i}}(x_\alpha)(1+\delta) \tag{4.11}$$

for $i=1,2,\ldots,k$ and $\alpha=1,2,\ldots,m$. These parameters are just the components of $r(x)$ for $x=x_\alpha$. This indicates that the poset is graded in the sense that if x is any element of the poset and $c=(x_1, x_2, \ldots, x_m)$ is any maximal chain in the poset

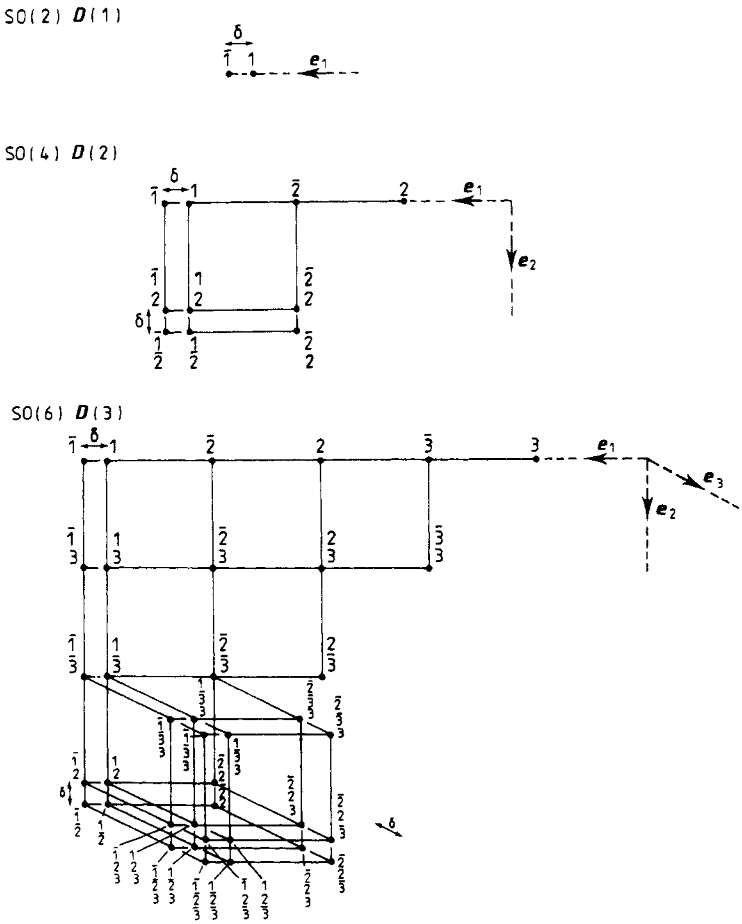


Figure 4. Fundamental posets $\boldsymbol{D}(k)$ of SO($2k$) for $k=1$, 2, 3.

containing $\boldsymbol{x}$ then $\boldsymbol{x}=\boldsymbol{x}_\alpha$ for

$$\alpha = m - a + 1 \qquad \text{with } a = \left[\sum_{i=1}^{k} r_i(\boldsymbol{x})\right]_{\delta=0} \tag{4.12}$$

where a is the grade of $\boldsymbol{x}$.

With this notation it is then convenient to introduce:

$$\mu_i(a) = [r_i(\boldsymbol{x})]_{\delta=0} = y_i(\boldsymbol{x}_\alpha) + y_{\bar{i}}(\boldsymbol{x}_\alpha), \tag{4.13}$$

so that

$$\sum_{i=1}^{k} \mu_i(a) = a \tag{4.14}$$

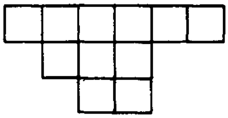
and, by virtue of (3.9)–(3.12),

$$\mu_i(a) > \mu_{i+1}(a) > 0 \qquad \text{for } i = 1, 2, \ldots, p-1 \tag{4.15}$$

with each $\mu_i(a)$ an integer for $i = 1, 2, \ldots, p$. Thus, associated with each poset element

$\boldsymbol{x}$ with a grading specified by a, there exists a partition $\mu(a) = (\mu_1(a), \mu_2(a), \ldots, \mu_p(a))$ of a into p unequal parts, where p is the length of the single column augmented standard Young tableaux constituting $\boldsymbol{x}$.

Each such partition $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ of m into p unequal parts defines a shifted Young diagram G^μ consisting of m boxes arranged in p rows of length $\mu_1, \mu_2, \ldots, \mu_m$ left adjusted to a diagonal line. Thus G^μ may be obtained from the conventional Young diagram F^μ by shifting the boxes in the ith row $i-1$ steps to the right. For example:

$G^{(632)} =$ 

Corresponding to each shifted Young diagram G^μ consisting of m boxes there exist g^μ restricted standard shifted Young tableaux Z^μ obtained by inserting the integers $1, 2, \ldots, m$ into the boxes of G^μ without repetition and in such a way that the entries increase across each row from left to right and down each column from top to bottom. Denoting the entry in the box of the ith row and jth column of G^μ by $\eta(i, j)$ then

$$\eta(i,j) = \eta(k,l) \qquad \text{if and only if } (i,j) = (k,l) \tag{4.16}$$

$$\eta(i,j) > \eta(i,j-1) \qquad \text{for } 1 \leqslant i < j \tag{4.17}$$

$$\eta(i,j) > \eta(i-1,j) \qquad \text{for } 2 \leqslant i \leqslant j \tag{4.18}$$

with

$$\eta(i,j) \in \{1, 2, \ldots, m\}. \tag{4.19}$$

The number of restricted standard shifted Young tableaux Z^μ has been determined by Thrall (1952) and may be expressed in the form (Macdonald 1979, p 135)

$$g^\mu = m! \prod_{1 \leqslant i \leqslant p} \frac{1}{\mu_i!} \prod_{1 \leqslant i < j \leqslant p} \left(\frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right) = \frac{m!}{J(\mu)}, \tag{4.20}$$

where $J(\mu)$ is the product of the supplemented hook lengths associated with each box of G^μ. The supplementation involves adding μ_{i+1} boxes to the ith column of G^μ for $i = 1, 2, \ldots, p-1$. For example

$$g^{(632)} = 11! \Bigg/ \begin{array}{cccccc} 9 & 8 & 6 & 5 & 2 & 1 \\ . & 5 & 3 & 2 & & \\ . & . & 2 & 1 & & \\ . & . & & & & \end{array} = 154$$

where the supplementation has been indicated by the inclusion of dots.

Returning to the maximal chains $\boldsymbol{c} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$ of each poset it is easy to see that for each possible $\boldsymbol{x}_1$, $\alpha = 1$ and the grade a of $\boldsymbol{x}_1$ is m where:

$$m = \begin{cases} \frac{1}{2}(k-1)(k+2) & \text{for SU}(k), \\ k(k+1) & \text{for SO}(2k+1), \\ k(k+1) & \text{for Sp}(2k), \\ k & \text{for SO}(2k). \end{cases} \tag{4.21}$$

Moreover,

$$\mu(m)=\begin{cases}(k,k-1,\ldots,2,1) & \text{for } \mathrm{SU}(k),\\ (2k,2k-2,\ldots,2) & \text{for } \mathrm{SO}(2k+1),\\ (2k,2k-2,\ldots,2) & \text{for } \mathrm{Sp}(2k),\\ (2k-1,2k-3,\ldots,1) & \text{for } \mathrm{SO}(2k).\end{cases} \tag{4.22}$$

Just as standard Young tableaux T^{λ} and $T^{\Delta;\lambda}$ have been generalised by introducing augmented standard Young tableaux $T^{\lambda}_{\varepsilon}$ and $T^{\Delta;\lambda}_{\varepsilon}$, so restricted standard shifted Young tableaux Z^{μ} may be generalised by introducing augmented restricted standard shifted Young tableaux Z^{μ}_{ε}. This augmentation, as before, involves replacing the entries $\eta(i,j)$ by $\eta(i,j)_{\varepsilon(i,j)}$ with $\varepsilon(i,j)=0,\bar{1}$ or 1.

For both SU(k) and Sp($2k$)

$$\varepsilon(i,j)=0, \tag{4.23}$$

whilst for SO($2k+1$)

$$\varepsilon(i,j)=\begin{cases}0 & \text{if } j\neq 2k-i+1\\ \bar{1} \text{ or } 1 & \text{if } j=2k-i+1\end{cases} \tag{4.24}$$

and for SO($2k$)

$$\varepsilon(i,j)=\begin{cases}0 & \text{if } j\neq 2k-i\\ \bar{1} \text{ or } 1 & \text{if } j=2k-i.\end{cases} \tag{4.25}$$

This augmentation is just sufficient to distinguish all possible distinct $\boldsymbol{x}_1$ in each maximal chain $\boldsymbol{c}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_m)$. The only entries of $Z^{\mu(m)}_{\varepsilon}$ which have $\varepsilon(i,j)\neq 0$ are those at the extreme right-hand end of each row, and then only in the case of SO($2k+1$) and SO($2k$). The entries for which $\varepsilon(i,j)=\bar{1}$ are in one-to-one correspondence with those $\boldsymbol{x}_1$ for which $r_i(\boldsymbol{x}_1)$, as given by (3.10) and (3.12), contains δ, and correspondingly with those Yamanouchi symbols (4.8) which contain $\bar{i}$, by virtue of (4.6).

It is clear that quite apart from this duplication of tableaux signified by the presence of $\bar{i}$ rather than i, the Yamanouchi symbols (4.8) serve to specify each standard restricted shifted Young tableaux $Z^{\mu(m)}$. This comes about because reading each sequence (4.8) from right to left gives the labels of the rows, i, of $Z^{\mu(m)}$ containing the entries $1,2,\ldots,m$ taken in turn. This is entirely analogous to the use of Yamanouchi symbols to specify standard restricted unshifted Young tableaux (Hamermesh 1962).

It thus follows that the maximal chains $\boldsymbol{c}$ of each poset under consideration are in one-to-one correspondence with the augmented restricted standard shifted Young tableaux $Z^{\mu(m)}_{\varepsilon}$, with the partitions $\mu(m)$ defined by (4.22) for each of the classical groups. The total number of such tableaux, and hence the total number of maximal chains of each poset, is given by

$$N(\boldsymbol{c})=\begin{cases}\dfrac{[\frac{1}{2}k(k+1)]!}{(2k-1)!}\displaystyle\prod_{i=1}^{k-1}\frac{i!}{(2i-1)!} & \text{for } \mathrm{SU}(k),\\ \dfrac{2^k[k(k+1)]!}{(2k)!}\displaystyle\prod_{i=1}^{k-1}\frac{i!}{(k+i)!} & \text{for } \mathrm{SO}(2k+1),\\ \dfrac{[k(k+1)]!}{(2k)!}\displaystyle\prod_{i=1}^{k-1}\frac{i!}{(k+i)!} & \text{for } \mathrm{Sp}(2k),\\ \dfrac{2^k[k^2]!}{(2k-1)!}\displaystyle\prod_{i=1}^{k-1}\frac{i!}{(k+i-1)!} & \text{for } \mathrm{SO}(2k).\end{cases} \tag{4.26}$$

The factors of 2^k appearing in (4.26) are a direct result of the fact that for both SO$(2k+1)$ and SO$(2k)$ there exist 2^k possible first elements $\boldsymbol{x}_1$ in maximal chains $\boldsymbol{c} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$. In fact $\boldsymbol{x}_1$ is an augmented standard Young tableau $T_\varepsilon^{(1^k)}$ consisting of a single column of length k. In each case the tableau $Z_\varepsilon^{\mu(m)}$ corresponding to $\boldsymbol{c}$ is characterised by a signature vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ with

$$\sigma_i = \begin{cases} 1 & \text{if } \varepsilon(i,j) = 0 \text{ or } 1 \text{ for all } j \\ -1 & \text{if } \varepsilon(i,j) = \bar{1} \text{ for some } j \end{cases} \tag{4.27}$$

for $i = 1, 2, \ldots, k$. The signature itself is then defined by

$$\sigma = \prod_{i=1}^{k} \sigma_i. \tag{4.28}$$

With this definition the signature σ of $Z_\varepsilon^{\mu(m)}$ coincides with the signature σ of $T_\varepsilon^{(1^k)} = \boldsymbol{x}_1$ defined by (2.19) if $\boldsymbol{x}_1$ is indeed the first element of the maximal chain $\boldsymbol{c}$ corresponding to $Z_\varepsilon^{\mu(m)}$.

These ideas on maximal chains $\boldsymbol{c}$, Yamanouchi symbols $Y(\boldsymbol{c})$ and augmented restricted standard Young tableaux $Z_\varepsilon^{\mu(m)}$ may be illustrated through a consideration of figure 5 in which are displayed the posets $\boldsymbol{C}(2)$ and $\boldsymbol{D}(2)$ associated with Sp(4) and SO(4), respectively. The corresponding graphs are labelled, and in each case a maximal chain $\boldsymbol{c}$ is indicated.

In the case of Sp(4) the maximal chain is defined by

$$\boldsymbol{c} = \begin{matrix} \bar{1} & \bar{1} & 1 & 1 & \bar{2} & 2 \\ \bar{2} & 2 & 2 & & & \end{matrix} \tag{4.29}$$

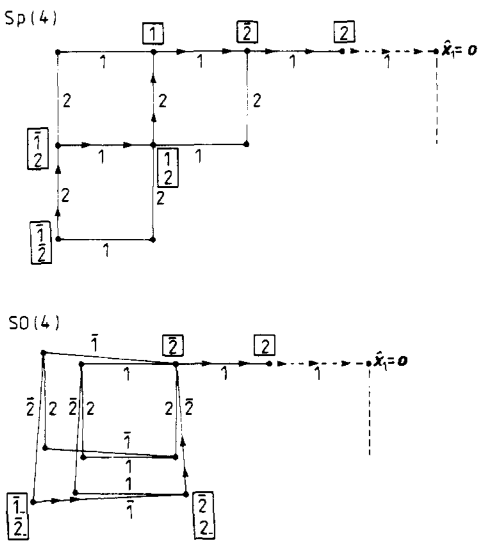


Figure 5. Examples of maximal chains for Sp(4) and SO(4).
For Sp(4) $\boldsymbol{c} = \begin{matrix} \bar{1} & \bar{1} & 1 & 1 & \bar{2} & 2 \\ \bar{2} & 2 & 2 \end{matrix}$, $Y(\boldsymbol{c}) = 2\ 1\ 2\ 1\ 1\ 1$, $\boldsymbol{d}(\boldsymbol{c}) = \begin{matrix} 1 \\ 2 \end{matrix}$
for SO(4) $\boldsymbol{c} = \begin{matrix} \bar{1}_- & \bar{2} & \bar{2} & 2 \\ \bar{2}_- & 2_- \end{matrix}$, $Y(\boldsymbol{c}) = \bar{1}\ \bar{2}\ 1\ 1$, $\boldsymbol{d}(\boldsymbol{c}) = \begin{matrix} \bar{1}_- & \bar{2} \\ \bar{2}_- & 2_- \end{matrix}$

for which

$$Y(\boldsymbol{c}) = 2\ 1\ 2\ 1\ 1\ 1 \qquad \text{and} \qquad Z_\varepsilon^{\mu(m)} = \begin{matrix} 1 & 2 & 3 & 5 \\ & 4 & 6 & \end{matrix} \tag{4.30}$$

with

$$\mu(m) = \mu(6) = (4, 2). \tag{4.31}$$

For SO(4) the maximal chain is defined by

$$\boldsymbol{c} = \begin{matrix} \bar{1}_- & \bar{2} & \bar{2} & 2 \\ \bar{2}_- & 2_- & & \end{matrix} \tag{4.32}$$

for which

$$Y(\boldsymbol{c}) = \bar{1}\ \bar{2}\ 1\ 1 \qquad \text{and} \qquad Z_\varepsilon^{\mu(m)} = \begin{matrix} 1 & 2 & \bar{4} \\ & \bar{3} & \end{matrix} \tag{4.33}$$

with

$$\mu(m) = \mu(4) = (3, 1). \tag{4.34}$$

In this case

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = (-1, -1) \quad \text{and} \quad \sigma = 1. \tag{4.35}$$

In (4.30) and (4.33) it has been convenient to write $\eta(i, j)_{\varepsilon(i,j)}$ as $\eta(i, j)$ for $\varepsilon(i, j) = 0$ or 1 and as $\overline{\eta(i, j)}$ for $\varepsilon(i, j) = \bar{1}$.

It is worth pointing out that each Yamanouchi symbol $Y(\boldsymbol{c})$, (4.8), defines a sequence of partitions $\mu(a)$ for $a = 1, 2, \ldots, m$ through (4.13). Each such partition $\mu(a)$ determines the shape of that part of the corresponding tableau $Z_\varepsilon^{\mu(m)}$ obtained by deleting all but the first a entries 1 or $\bar{1}$, 2 or $\bar{2} \ldots a$ or $\bar{a}$.

In examples (4.30) and (4.33) these sequences of partitions are therefore: (1), (2), (3), (3, 1), (4, 1), (4, 2) and (1), (2), (2, 1), (3, 1) respectively.

5. Generating functions

The augmented restricted standard shifted Young tableaux are of course simply a combinatorial device for enumerating all possible maximal chains $\boldsymbol{c}$ in each fundamental poset. The maximal chains define through the juxtaposition of their elements $\boldsymbol{x}_1$, $\boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ those augmented standard Young tableaux whose columns are all distinct and whose first row has length m. To obtain all possible augmented standard Young tableaux it is necessary to generate the corresponding multichains $\boldsymbol{m}$.

Each maximal chain $\boldsymbol{c}$ is not only devoid of repetitions amongst its elements but also unrefinable in the sense that there is no element $\boldsymbol{z}$ satisfying (4.3) or (4.4). In the case of a multichain $\boldsymbol{m}$ both of these conditions may be violated. However, as pointed out by Baclawski (1983) in the case of the posets of SU(k) and Sp($2k$), associated with each multichain $\boldsymbol{m}$ there exists a unique maximal chain $\boldsymbol{c}$. In generalising this result to the posets of SO($2k+1$) and Sp($2k$) it is necessary to overcome a slight complication: namely that of the lack of uniqueness of the first element $\boldsymbol{x}_1$ of a maximal chain $\boldsymbol{c} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$. For both SU($k$) and Sp($2k$) this element $\boldsymbol{x}_1$ is unique, but for both SO($2k+1$) and Sp($2k$) there are a total of 2^k possible first elements $\boldsymbol{x}_1$ of grade m.

In order to cope with this difficulty it is worthwhile bounding the posets by the introduction not only of a greatest element $\hat{\boldsymbol{x}}_1$ as in (4.1) but also of a least element $\hat{\boldsymbol{x}}_0$ such that

$$\hat{\boldsymbol{x}}_0 < \boldsymbol{x} \qquad \text{for all } \boldsymbol{x} \in \boldsymbol{P}(k). \tag{5.1}$$

The position vector of this element is not important but it is important to label the edges joining $\hat{\boldsymbol{x}}_0$ to each element $\boldsymbol{x}_1$ of grade m by

$$\lambda(\hat{\boldsymbol{x}}_0, \boldsymbol{x}_1) = 0. \tag{5.2}$$

Corresponding to each maximal chain $\boldsymbol{c} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$ there exists a descent subchain $\boldsymbol{d}(\boldsymbol{c})$ consisting of those elements $\boldsymbol{x}_\alpha$ of $\boldsymbol{c}$ such that

$$\lambda(\boldsymbol{x}_{\alpha-1}, \boldsymbol{x}_\alpha) = Y_{\alpha-1}(\boldsymbol{c}) < Y_\alpha(\boldsymbol{c}) = \lambda(\boldsymbol{x}_\alpha, \boldsymbol{x}_{\alpha+1}) \tag{5.3}$$

for $\alpha = 1, 2, \ldots, m$ with $\boldsymbol{x}_0 = \hat{\boldsymbol{x}}_0$ and $\boldsymbol{x}_{m+1} = \hat{\boldsymbol{x}}_1$. The ordering imposed on the values of the labels $\lambda(\boldsymbol{x}, \boldsymbol{y})$ is defined, not by (2.2), but by:

$$\bar{k} > \ldots > \bar{2} > \bar{1} > 0 > k > \ldots > 2 > 1. \tag{5.4}$$

The descent set of $\boldsymbol{c}$ is then defined by

$$D(\boldsymbol{c}) = \{a \colon a = m - \alpha + 1 \text{ with } Y_{\alpha-1}(\boldsymbol{c}) < Y_\alpha(\boldsymbol{c})\}. \tag{5.5}$$

Extending the Yamanouchi symbol $Y(\boldsymbol{c})$ of (4.8) by the introduction of a zeroth component $Y_0(\boldsymbol{c}) = 0$, the examples (4.29) and (4.32) yield by way of illustration

$$Y(\boldsymbol{c}) = 0\ 2\ 1\ 2\ 1\ 1\ 1 \qquad D(\boldsymbol{c}) = \{4\} \qquad \boldsymbol{d}(\boldsymbol{c}) = \begin{array}{l} 1 \\ 2 \end{array} \tag{5.6}$$

and

$$Y(\boldsymbol{c}) = 0\ \bar{1}\ \bar{2}\ 1\ 1 \qquad D(\boldsymbol{c}) = \{3, 4\} \qquad \boldsymbol{d}(\boldsymbol{c}) = \begin{array}{ll} \bar{1}_- & \bar{2} \\ \bar{2}_- & 2_- \end{array} \tag{5.7}$$

respectively.

The same descent set (5.5) may be identified directly from the corresponding tableaux $Z_\varepsilon^{(m)}$ by noting that a is in $D(\boldsymbol{c})$ if and only if either a is in the ith row and $a+1$ is in the jth row for some $j < i$, or $\bar{a}$ is in the ith row and $\overline{a+1}$ is not in the jth row for any $j > i$.

Thus for example from (4.30) and (4.33) the tableaux

$$Z_\varepsilon^{\mu(m)} = \begin{array}{llll} 1 & 2 & 3 & 5 \\ 4 & 6 & & \end{array} \qquad \text{and} \qquad Z_\varepsilon^{\mu(m)} = \begin{array}{lll} 1 & 2 & \bar{4} \\ \bar{3} & & \end{array} \tag{5.8}$$

indicate once more that

$$D(\boldsymbol{c}) = \{4\} \qquad \text{and} \qquad D(\boldsymbol{c}) = \{3, 4\} \tag{5.9}$$

respectively, as in (5.6) and (5.7).

The maximal chain $\boldsymbol{c}$ corresponding to the multichain $\boldsymbol{m} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q)$ is constructed by including in $\boldsymbol{c}$ all the distinct elements of $\boldsymbol{m}$ and interpolating additional elements, if required, in the form of the unique unrefinable subchains between successive elements $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q, \boldsymbol{x}_{q+1}$ with $\boldsymbol{x}_0 = \hat{\boldsymbol{x}}_0$ and $\boldsymbol{x}_{q+1} = \hat{\boldsymbol{x}}_1$. Such an interpolation is necessary between $\boldsymbol{x}_j$ and $\boldsymbol{x}_{j+1}$ and $j = 0, 1, 2, \ldots, q$ if there exists any $\boldsymbol{z}$ in the poset such that $\boldsymbol{x}_j < \boldsymbol{z} < \boldsymbol{x}_{j+1}$. The subchain $\boldsymbol{s}_j = (\boldsymbol{x}_\alpha, \boldsymbol{x}_{\alpha+1}, \ldots, \boldsymbol{x}_\beta)$ interpolated between $\boldsymbol{x}_j$

and $\boldsymbol{x}_{j+1}$ is the unique subchain satisfying the conditions:

$$\lambda(\boldsymbol{x}_{j-1}, \boldsymbol{x}_\gamma) = Y_{\gamma-1} \geqslant Y_\gamma = \lambda(\boldsymbol{x}_\gamma, \boldsymbol{x}_{\gamma+1}) \tag{5.10}$$

for $\gamma = \alpha, \alpha+1, \ldots, \beta$ with $\boldsymbol{x}_{\alpha-1} = \boldsymbol{x}_j$ and $\boldsymbol{x}_{\beta+1} = \boldsymbol{x}_{j+1}$.

For $j > 0$ the uniqueness of this subchain is established by making use of the fact that (4.11) applied to $r_i(\boldsymbol{x}_j) - r_i(\boldsymbol{x}_{j+1})$ determines the number of the labels Y_γ equal to $\bar{i}$ or i. Arranging all such labels in descending order, as determined by (5.4), gives the required sequence

$$Y_{\alpha-1} Y_\alpha Y_{\alpha+1} \ldots Y_\beta, \tag{5.11}$$

which fixes the subchain $\boldsymbol{s}_j$ completely.

For $j = 0$ interpolation is only necessary if the first element $\boldsymbol{x}_1$ of the multichain $\boldsymbol{m}$ is not of grade m. In this case the first interpolated element $\boldsymbol{x}_\alpha$ of $\boldsymbol{s}_0 = (\boldsymbol{x}_\alpha, \boldsymbol{x}_{\alpha+1}, \ldots, \boldsymbol{x}_\beta)$ must be chosen to be the unique element of grade m such that the application of (4.11) to $r_i(\boldsymbol{x}_\alpha) - r_i(\boldsymbol{x}_1)$ indicates that the number of the labels Y_α equal to $\bar{i}$ is zero for all i. Arranging the other labels in descending order immediately after the initial label $Y_{\alpha-1} = \lambda(\hat{\boldsymbol{x}}_0, \boldsymbol{x}_\alpha) = 0$ then yields the required sequence (5.11) with

$$Y_{\alpha-1} > Y_\alpha > Y_{\alpha+1} > \ldots > Y_\beta, \tag{5.12}$$

in accordance with (5.4). The uniqueness of the corresponding subchain $\boldsymbol{s}_0$ then follows from the uniqueness of the element $\boldsymbol{x}_\alpha$ of grade m satisfying the conditions determined by $\boldsymbol{x}_1$.

To exemplify this construction procedure recourse can be made to figure 5 once again. In the case of Sp(4) the multichain

$$\boldsymbol{m} = \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & \end{matrix} \tag{5.13}$$

requires interpolation before $\begin{smallmatrix}1\\2\end{smallmatrix}$ and between $\begin{smallmatrix}1\\2\end{smallmatrix}$ and 2. In the first case the Yamanouchi sequence (5.11) is 0 2 1 and in the second 2 1 1, so that

$$\boldsymbol{s}_0 = \begin{matrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \end{matrix} \quad \text{and} \quad \boldsymbol{s}_3 = 1 \;\; \bar{2}. \tag{5.14}$$

Interpolating these between the distinct elements of $\boldsymbol{m}$ yields the required maximal chain $\boldsymbol{c}$ of (4.29) with descent subchain $\boldsymbol{d}(\boldsymbol{c})$ given by (5.6).

Similarly in the case of SO(4) the multichains

$$\boldsymbol{m} = \begin{matrix} \bar{1}_- & \bar{1}_- & 2 & 2 & 2 \\ \bar{2}_- & \bar{2}_- & & & \end{matrix} \quad \text{and} \quad \boldsymbol{m} = \bar{2} \;\; \bar{2} \;\; 2 \tag{5.15}$$

require interpolation between $\begin{smallmatrix}\bar{1}_-\\\bar{2}_-\end{smallmatrix}$ and 2 and before $\bar{2}$, respectively. The Yamanouchi sequences (5.11) are

$$0 \;\; \bar{2} \;\; 1 \;\; 1 \text{ with } \boldsymbol{s}_2 = \bar{1} \;\; \bar{2} \quad \text{and} \quad 0 \;\; 2 \;\; 1 \;\; 1 \text{ with } \boldsymbol{s}_0 = \begin{matrix} 1 & 1 & \bar{2}. \\ 2 & & \end{matrix} \tag{5.16}$$

The corresponding maximal chains are

$$\boldsymbol{c} = \begin{matrix} \bar{1}_- & \bar{1} & \bar{2} & 2 \\ \bar{2}_- & & & \end{matrix} \quad \text{and} \quad \boldsymbol{c} = \begin{matrix} 1 & 1 & \bar{2} & 2 \\ 2 & & & \end{matrix} \tag{5.17}$$

with descent subchains

$$d(c) = \begin{matrix} \bar{1}_- \\ 2_- \end{matrix} \qquad \text{and} \qquad d(c) = -. \tag{5.18}$$

It is no accident that the descent subchain $d(c)$ of the maximal chain c obtained from a multichain m is a subchain of m in each of the above examples. This is a direct consequence of (5.10) which ensures that no interpolated element belongs to $d(c)$. It then follows that each multichain m may be generated through appropriate repetitions of certain selected elements of the corresponding maximal chain c including every element of $d(c)$ at least once. Hence

$$\sum_{m} \prod_{x_i \in m} x_i = \sum_{c} \left\{ \prod_{x_j \in d(c)} x_j \Big/ \prod_{x_j \in c} (1 - x_i) \right\}, \tag{5.19}$$

where of course

$$1/(1 - x_i) = \sum_{n_i = 0}^{\infty} x_i^{n_i}, \tag{5.20}$$

and the multichains m are formed by arranging the elements of each summand in increasing order.

In order to write down the corresponding character generator, (1.2), it is merely necessary to make use of (1.7) and the one-to-one correspondence between augmented standard Young tableaux and multichains, together with a suitable labelling scheme, this time for the poset elements themselves.

Each poset element x takes the form of a single column augmented standard Young tableau. Thus

$$x = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{matrix} \qquad \text{with } x_i = \eta(i, 1)_{\varepsilon(i,1)} \qquad \text{for } i = 1, 2, \ldots, p. \tag{5.21}$$

Such an element has a well defined signature, σ, and grade, a, defined by (2.13) and (4.12) respectively. With this notation the appropriate label may be written in the form:

$$l(x) = \left\{ \prod_{i=1}^{p} \alpha_i(\sigma) t_i(\phi) \right\}^{s}, \tag{5.22}$$

where

$$s = \begin{cases} 1 & \text{if } 1 \leqslant a \leqslant m & \text{for SU}(k) \text{ and Sp}(2k) \\ 1 & \text{if } 1 \leqslant a < m & \text{for SO}(2k+1) \text{ and SO}(2k) \\ \frac{1}{2} & \text{if } a = m & \text{for SO}(2k+1) \text{ and SO}(2k), \end{cases} \tag{5.23}$$

$$\alpha_i(\sigma) = \begin{cases} \alpha_i & \text{if } 1 \leqslant i < k & \text{for SU}(k) \\ \alpha_i & \text{if } 1 \leqslant i \leqslant k & \text{for SO}(2k+1) \text{ and Sp}(2k) \\ \alpha_i & \text{if } 1 \leqslant i < k & \text{for SO}(2k) \\ \alpha_k^{\sigma} & \text{if } i = k & \text{for SO}(2k), \end{cases} \tag{5.24}$$

and finally,

$$t_i(\boldsymbol{\phi}) = \begin{cases} \exp(\mathrm{i}\phi_j) & \text{if } \eta(i,1) = j \\ \exp(-\mathrm{i}\phi_j) & \text{if } \eta(i,1) = \bar{j} \\ 1 & \text{if } \eta(i,1) = 0. \end{cases} \tag{5.25}$$

The need for the exponent s with value $\frac{1}{2}$ in the case of $SO(2k+1)$ and $SO(2k)$ is a consequence of the generalisation of multichains $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q)$ to include those of the form $(\boldsymbol{x}_1^{1/2}, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{q+1})$. This is itself necessitated by the need to generate not only the tableaux T_ε^λ but also $T_\varepsilon^{\Delta;\lambda}$. Similarly the need to incorporate the signature, $\sigma = \pm 1$, is a direct consequence of (2.17) and (2.19), which indicate those tableaux associated with the irreducible representations $[\lambda]_\pm$ and $[\Delta; \lambda]_\pm$ of $SO(2k)$.

Incorporating this labelling (5.22) in (5.19) yields the required character generator (1.2) in the form:

$$\begin{aligned} X_G(\boldsymbol{c}, \boldsymbol{\phi}) &= \sum_{\lambda_G} \chi^{\lambda_G}(\boldsymbol{\phi})\boldsymbol{\alpha}^{\boldsymbol{\lambda}} \\ &= \sum_{\boldsymbol{mn}} \prod_{\boldsymbol{x}_i \in \boldsymbol{m}} l(\boldsymbol{x}_i) \\ &= \sum_{\boldsymbol{c}} \left\{ \prod_{\boldsymbol{x}_j \in \boldsymbol{d}(\boldsymbol{c})} l(\boldsymbol{x}_j) \Big/ \prod_{\boldsymbol{x}_i \in \boldsymbol{c}} (1 - l(\boldsymbol{x}_i)) \right\}. \end{aligned} \tag{5.26}$$

Whilst this formula gives the character generator it is perhaps worth recasting in a form which enables all the terms of the formula to be written down by inspection of the corresponding set of augmented restricted standard Young tableaux $Z_\varepsilon^{\mu(m)}$. The appropriate labels of the poset elements $\boldsymbol{x}$ are then slightly different. In fact setting

$$l(\boldsymbol{x}) = \Gamma(\mu(a)) \tag{5.27}$$

where $\boldsymbol{x}$ has grade a and the partition $\mu(a)$ is determined by (4.13), the character generator may be written in the form

$$\begin{aligned} X_G(\boldsymbol{\alpha}, \boldsymbol{\phi}) &= F_{\mu(m)}(\boldsymbol{A}, \boldsymbol{X}) \\ &= \sum_{Z_\varepsilon^{\mu(m)}} \left\{ \prod_{a \in D(\boldsymbol{c})} \Gamma(\mu(a)) \Big/ \prod_{a=1}^{m} [1 - \Gamma(\mu(a))] \right\} \end{aligned} \tag{5.28}$$

with

$$\Gamma(\mu(a)) = \left\{ A_p(\sigma) \prod_{i=1}^{p} X_{\mu_i(a)}(\sigma_i) \right\}^s. \tag{5.29}$$

The notation is such that:

$$A_p(\sigma) = \prod_{i=1}^{p} \alpha_i(\sigma), \tag{5.30}$$

with $\alpha_i(\sigma)$ defined by (5.24), whilst s is given by (5.23), and the signature vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is determined from the entries in $Z_\varepsilon^{\mu(m)}$ by (4.27). For both $SU(k)$ and $Sp(2k)$ the signature vector has no role to play since augmentation of the tableaux is unnecessary. Hence for these groups

$$X_{\mu_i(a)}(\sigma_i) = X_{\mu_i(a)}. \tag{5.31}$$

However, for both SO($2k+1$) and SO($2k$)

$$X_{\mu_i(a)}(\sigma_i) = \begin{cases} X_{\mu_i(a)+1} & \text{if } \sigma_i = -1,\ \mu_i(m) = \mu_i(a) = \mu_{i-1}(a) - 2 \text{ (for } i > 1) \\ X_{\mu_i(a)} & \text{otherwise.} \end{cases} \tag{5.32}$$

Finally

$$\boldsymbol{X} = \begin{cases} (e^{i\phi_k}, e^{i\phi_{k-1}}, \ldots, e^{i\phi_2}, e^{i\phi_1}) & \text{for SU}(k) \\ (1, e^{i\phi_k}, e^{-i\phi_k}, \ldots, e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1}) & \text{for SO}(2k+1) \\ (e^{i\phi_k}, e^{-i\phi_k}, \ldots, e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1}) & \text{for Sp}(2k) \\ (e^{i\phi_k}, e^{-i\phi_k}, \ldots, e^{i\phi_2}, e^{-i\phi_2}, e^{i\phi_1}, e^{-i\phi_1}) & \text{for SO}(2k). \end{cases} \tag{5.33}$$

This is the culmination of the programme initiated by Stanley (1980) who introduced standard restricted shifted Young tableaux in this context.

6. Discussion and exemplification

The complexities in the analysis of the previous sections are almost entirely occasioned by the need to introduce augmented tableaux in dealing with SO($2k+1$) and SO($2k$). Nonetheless it is remarkable that these groups can be encompassed both by the formula (5.26), first applied to SU(k) and Sp($2k$) by Baclawski (1983), and by the formula (5.28), first applied to SU(k) by Stanley (1980) and later to Sp($2k$) (King 1981).

Whilst (5.26) is easier to use for low values of k, as k increases it rapidly becomes difficult to enumerate all possible maximal chains $\boldsymbol{c}$ and to determine the descent subchains $\boldsymbol{d}(\boldsymbol{c})$. For this reason the use of augmented restricted standard shifted Young tableaux in (5.28) is to be preferred in most calculations. Moreover, it lends itself to computer implementation through the straightforward enumeration of such tableaux.

This task is left to other authors. Here we content ourselves with exhibiting the results appropriate to low values of k. The number of terms in each character generator is determined by the formula (4.26) for $N(\boldsymbol{c})$, the total number of maximal chains in each poset. The number of factors in the denominator of each term is given by the formula (4.21) for m, the length of each maximal chain. The manner in which m, $N(\boldsymbol{c})$ and $|\boldsymbol{P}(k)|$, the number of elements in each poset, increase with k is given in table 1 which extends the similar tabulation of Baclawski (1983).

Selecting those cases for which $N(\boldsymbol{c}) < 100$ the corresponding classical group character generators may be written down by inspection of the information displayed in table 2. In this table each maximal chain $\boldsymbol{c}$ has been displayed as a standard Young tableau T^λ or $T^{\Delta;\lambda}$ along with the corresponding descent subchain $\boldsymbol{d}(\boldsymbol{c})$, shown in the same way. These tableaux, which are not augmented, define the character generators completely. This can be seen from (5.21)–(5.26) since the only role of subscript $\varepsilon(i, 1)$ is to determine the signature σ. For all groups except SO($2k$), $\sigma = +1$, and for SO($2k$) for all columns of length p, with $p < k$, $\sigma = +1$, whilst for all columns of length k, $\sigma = +1$ or -1, according as the number of entries in the first column of $T^{\Delta;\lambda}$ given by $\bar{j}$, for some j, is even or odd respectively. It should be noted that the columns for which $s = \frac{1}{2}$ have been indicated by the use of $T^{\Delta;\lambda}$ rather than T^λ with entries in half boxes placed to the left of the symbol /.

Table 1. Data on low rank posets.

G	$\|\boldsymbol{P}(k)\|$	$N(\boldsymbol{c})$	m	References
SU(2)	2	1	2	1, 5
SU(3)	6	2	5	1, 2, 4, 6
SU(4)	14	12	9	2, 4
SU(5)	30	286	14	4
SO(3)	3	2	2	1, 5
SO(5)	15	20	6	1, 6
SO(7)	63	3696	12	6
SO(9)	255	159 629 184	20	
Sp(2)	2	1	2	1, 5
Sp(4)	9	5	6	3, 4
Sp(6)	34	462	12	
Sp(8)	125	9 976 824	20	
SO(2)	2	2	1	
SO(4)	10	8	4	5
SO(6)	41	336	9	
SO(8)	162	384 384	16	

References: 1 Patera and Sharp (1979) 2 Stanley (1980) 3 King (1981) 4 Baclawski (1983) 5 King and El-Sharkaway (1982) 6 Gaskell (1983)

Table 2. Maximal chains and descent subchains.

G	$Z_\varepsilon^{\mu(m)}$	$\boldsymbol{d}(\boldsymbol{c})/\boldsymbol{c}$	σ
SU(2)	12	— / 1 2	1
SU(3)	123 45	— / 1 1 1 2 3 / 2 3	1
	124 35	2 / 1 1 2 2 3 3 / 2 3 3	1
SU(4)	1234 567 89	— / 1 1 1 1 1 1 2 3 4 / 2 2 2 3 4 / 3 4	1
	1234 568 79	1 / 1 1 1 1 1 1 2 3 4 3 / 2 2 3 3 4 4 / 3 4 4	1
	1235 467 89	2 / 1 1 1 1 1 2 2 3 4 4 / 2 2 2 3 4 4 / 3 4	1
	1235 468 79	1 2 / 1 1 1 1 1 2 2 3 4 3 4 / 2 2 3 3 4 4 4 / 3 4 4	1
	1236 457 89	2 / 1 1 1 1 2 2 2 3 4 3 / 2 2 2 3 3 4 / 3 4	1

Table 2. (continued)

G	$Z_{\varepsilon}^{\mu(m)}$	$d(c)/c$	σ
	1236 458 79	1 2 / 1 1 1 1 2 2 2 3 4 3 3 / 2 2 3 3 3 4 4 / 3 4 4	1
	1237 458 69	2 / 1 1 1 2 2 2 2 2 3 4 3 / 2 2 3 3 3 4 4 / 3 4 4 4	1
	1245 367 89	3 / 1 1 1 1 1 2 3 3 4 4 / 2 2 2 3 4 4 4 / 3 4	1
	1245 368 79	1 3 / 1 1 1 1 1 2 3 3 4 3 4 / 2 2 3 3 4 4 4 4 / 3 4 4	1
	1246 357 89	2 3 / 1 1 1 1 2 2 3 3 4 3 4 / 2 2 2 3 3 4 4 / 3 4	1
	1246 358 79	1 2 3 / 1 1 1 1 2 2 3 3 4 3 3 4 / 2 2 3 3 3 4 4 4 / 3 4 4	1
	1247 358 69	2 3 / 1 1 1 2 2 2 3 3 4 3 4 / 2 2 3 3 3 4 4 4 / 3 4 4 4	1
SO(3)	12	— / 1 / 0	1
	1$\bar{2}$	$\bar{1}$ / / $\bar{1}$ / 0	1
SO(5)	1234 56	— / 1 / 1 1 $\bar{2}$ 2 0 / 2 / 0	1
	123$\bar{4}$ 56	$\bar{1}$ / $\bar{1}$ / $\bar{1}$ $\bar{1}$ $\bar{2}$ 2 0 / 2 / 0	1
	1234 5$\bar{6}$	1 / / 1 / 1 1 $\bar{2}$ 2 0 $\bar{2}$ / / $\bar{2}$ / 0	1
	123$\bar{4}$ 5$\bar{6}$	$\bar{1}$ / $\bar{1}$ / $\bar{1}$ / $\bar{1}$ $\bar{1}$ $\bar{2}$ 2 0 $\bar{2}$ / / $\bar{2}$ / 0	1
	1235 46	$\bar{2}$ / 1 / 1 $\bar{2}$ $\bar{2}$ 2 0 0 / 2 / 0 0	1
	123$\bar{5}$ 46	$\bar{1}$ / $\bar{1}$ / $\bar{1}$ $\bar{2}$ $\bar{2}$ 2 0 0 / 2 / 0 0	1
	1235 4$\bar{6}$	1 / $\bar{2}$ / 1 / 1 $\bar{2}$ $\bar{2}$ 2 0 $\bar{2}$ / 0 / $\bar{2}$ / 0 0	1
	123$\bar{5}$ 4$\bar{6}$	$\bar{1}$ / / $\bar{1}$ / $\bar{1}$ $\bar{2}$ $\bar{2}$ 2 0 $\bar{2}$ / / $\bar{2}$ / 0 0	1
	1245 36	2 / 1 / 1 $\bar{2}$ 2 2 0 0 / 2 / 0 0 0	1
	124$\bar{5}$ 36	$\bar{1}$ 2 / $\bar{1}$ / $\bar{1}$ $\bar{2}$ 2 2 0 0 0 / 2 / 0 0 0	1

Table 2. (continued)

G	$Z_\varepsilon^{\mu(m)}$	$d(c)/c$	σ
	1245 $3\bar{6}$	$1 / 2 / 1 / 1\ \bar{2}\ 2\ 2\ 0$ $\bar{2} / 0 / \bar{2} / 0\ 0\ 0$	1
	$124\bar{5}$ $3\bar{6}$	$\bar{1} / 2 / \bar{1} / \bar{1}\ \bar{2}\ 2\ 2\ 0$ $\bar{2} / 0 / \bar{2} / 0\ 0\ 0$	1
	1236 45	$\bar{2} / 1 / \bar{2}\ \bar{2}\ \bar{2}\ 2\ 0$ $2 / 2 / 2\ 0$	1
	$123\bar{6}$ 45	$\bar{1} / / \bar{1} / \bar{2}\ \bar{2}\ \bar{2}\ 2\ 0$ $2 / / 2 / 2\ 0$	1
	1236 $4\bar{5}$	$\bar{2} / 1 / \bar{2}\ \bar{2}\ \bar{2}\ 2\ 0$ $2 / \bar{2} / 2\ 0$	1
	$123\bar{6}$ $4\bar{5}$	$\bar{1} / \bar{2} / \bar{1} / \bar{2}\ \bar{2}\ \bar{2}\ 2\ 0$ $\bar{2} / 2 / \bar{2} / 2\ 0$	1
	1246 35	$\bar{2}\ 2 / 1 / \bar{2}\ \bar{2}\ 2\ 2\ 0$ $2\ 0 / 2 / 2\ 0\ 0$	1
	$124\bar{6}$ 35	$\bar{1} / 2 / \bar{1} / \bar{2}\ \bar{2}\ 2\ 2\ 0$ $2 / 0 / 2 / 2\ 0\ 0$	1
	1246 $3\bar{5}$	$\bar{2}\ 2 / 1 / \bar{2}\ \bar{2}\ 2\ 2\ 0$ $2\ 0 / \bar{2} / 2\ 0\ 0$	1
	$124\bar{6}$ $3\bar{5}$	$\bar{1} / \bar{2}\ 2 / \bar{1} / \bar{2}\ \bar{2}\ 2\ 2\ 0$ $\bar{2} / 2\ 0 / \bar{2} / 2\ 0\ 0$	1
Sp(2)	12	$— / \bar{1}\ 1$	1
Sp(4)	1234 56	$— / \bar{1}\ \bar{1}\ \bar{1}\ 1\ \bar{2}\ 2$ $/ \bar{2}\ 2$	1
	1235 46	$1 / \bar{1}\ \bar{1}\ 1\ 1\ \bar{2}\ 2$ $2 / \bar{2}\ 2\ 2$	1
	1245 36	$\bar{2} / \bar{1}\ \bar{1}\ 1\ \bar{2}\ \bar{2}\ 2$ $2 / \bar{2}\ 2\ 2\ 2$	1
	1236 45	$1 / \bar{1}\ 1\ 1\ 1\ \bar{2}\ 2$ $\bar{2} / \bar{2}\ \bar{2}\ 2$	1
	1246 35	$1\ \bar{2} / \bar{1}\ 1\ 1\ \bar{2}\ \bar{2}\ 2$ $\bar{2}\ 2 / \bar{2}\ \bar{2}\ 2\ 2$	1
SO(2)	1	$— / 1 /$	+1
	$\bar{1}$	$\bar{1} / / \bar{1} /$	−1
SO(4)	123 4	$— / 1 / 1\ \bar{2}\ 2$ $/ 2 /$	+1
	$12\bar{3}$ 4	$\bar{1} / \bar{1} / \bar{1}\ \bar{2}\ 2$ $/ 2 /$	−1
	123 $\bar{4}$	$1 / / 1 / 1\ \bar{2}\ 2$ $\bar{2} / / \bar{2} /$	−1
	$12\bar{3}$ $\bar{4}$	$\bar{1} / / \bar{1} / \bar{1}\ \bar{2}\ 2$ $\bar{2} / / \bar{2} /$	+1

Table 2. (continued)

G	$Z_\varepsilon^{\mu(m)}$	$d(c)/c$	σ
	124 3	$\bar{2}\ /1\ /\ \bar{2}\ \bar{2}\ 2$ $2/\ 2\ /\ 2$	+1
	$12\bar{4}$ 3	$\bar{1}/\ /\bar{1}\ /\ \bar{2}\ \bar{2}\ 2$ $2//\ 2\ /\ 2$	−1
	124 $\bar{3}$	$\bar{2}\ /1\ /\ \bar{2}\ \bar{2}\ 2$ $2/\ \bar{2}\ /\ 2$	−1
	$12\bar{4}$ $\bar{3}$	$\bar{1}\ /\ \bar{2}\ /\bar{1}\ /\ \bar{2}\ \bar{2}\ 2$ $\bar{2}\ /\ 2/\ \bar{2}\ /\ 2$	+1

To illustrate the ease with which the contribution to the character generator may be written down for each term displayed the following examples should suffice.

$$\begin{array}{l} 1\ 2\ /1\ 1\ 1\ 1\ 1\ 2\ 2\ 3\ 4 \\ 3\ 4/\ 2\ 2\ 3\ 3\ 4\ 4 \\ 4\ \ \ /\ 3\ 4\ 4 \end{array}$$

$$\begin{aligned} &\Rightarrow \alpha_1\alpha_2\alpha_3 \exp \mathrm{i}(\phi_1+\phi_3+\phi_4)\alpha_1\alpha_2 \exp \mathrm{i}(\phi_2+\phi_4) \\ &\quad \times 1/\{[1-\alpha_1\alpha_2\alpha_3 \exp \mathrm{i}(\phi_1+\phi_2+\phi_3)][1-\alpha_1\alpha_2\alpha_3 \exp \mathrm{i}(\phi_1+\phi_2+\phi_4)] \\ &\quad \times [1-\alpha_1\alpha_2\alpha_3 \exp \mathrm{i}(\phi_1+\phi_3+\phi_4)][1-\alpha_1\alpha_2 \exp \mathrm{i}(\phi_1+\phi_3)] \\ &\quad \times [1-\alpha_1\alpha_2 \exp \mathrm{i}(\phi_1+\phi_4)][1-\alpha_1\alpha_2 \exp \mathrm{i}(\phi_2+\phi_4)] \\ &\quad \times [1-\alpha_1 \exp \mathrm{i}\phi_2][1-\alpha_1 \exp \mathrm{i}\phi_3][1-\alpha_1 \exp \mathrm{i}\phi]\} \end{aligned} \tag{6.1}$$

$$\begin{array}{l} \bar{1}/\ /\bar{1}\ /\ \bar{2}\ \bar{2}\ 2 \\ 2//\ 2\ /\ 2 \end{array}$$

$$\begin{aligned} &\Rightarrow \alpha_1^{1/2}\alpha_2^{-1/2} \exp \mathrm{i}\tfrac{1}{2}(-\phi_1+\phi_2)/[1-\alpha_1^{1/2}\alpha_2^{-1/2} \exp \mathrm{i}\tfrac{1}{2}(-\phi_1+\phi_2)] \\ &\quad \times [1-\alpha_1\alpha_2^{-1} \exp \mathrm{i}(-\phi_2+\phi_2)][1-\alpha_1 \exp(-\mathrm{i}\phi_2)][1-\alpha_1 \exp(\mathrm{i}\phi_2)] \end{aligned} \tag{6.2}$$

where of course the second factor in the denominator can be simplified to $(1-\alpha_1\alpha_2^{-1})$.

All other terms may be dealt with in the same way. For these low rank cases the results are not of course new. In table 1 reference has been made to those character generators which are already known.

It is perhaps worth pointing out the rather trivial result appropriate to SO(2), namely

$$\begin{aligned} X_{\mathrm{SO}(2)}(\alpha, \phi) = {} & 1/[1-\alpha^{1/2} \exp(\mathrm{i}\phi/2)] \\ & + \alpha^{-1/2} \exp(-\mathrm{i}\phi/2)/[1-\alpha^{-1/2} \exp(-\mathrm{i}\phi/2)]. \end{aligned} \tag{6.3}$$

This can be rewritten as

$$X_{\mathrm{SO}(2)}(\alpha, \phi) = (1-\alpha^{1/2}\alpha^{-1/2})/[1-\alpha^{1/2} \exp(\mathrm{i}\phi/2)][1-\alpha^{-1/2} \exp(-\mathrm{i}\phi/2)] \tag{6.4}$$

where it is important to note that $\alpha^{1/2}\alpha^{-1/2} \neq 1$ since α and α^{-1} should be thought of as two independent generating parameters.

The other point to note is that the local isomorphisms $Sp(2) \simeq SO(3) \simeq SU(2)$, $SO(5) \simeq Sp(4)$, $SO(4) \simeq SU(2) \times SU(2)$ and $SO(6) \simeq SU(4)$ enable a choice of generating function to be made. That choice with the fewest terms is clearly the most desirable. The numbers appearing in table 1 may be used to make this choice. They make it clear that the only new character generator which it is feasible to write down is that for Sp(6). This may be done, for example, from the poset diagram in figure 3, aided by the enumeration of the 462 tableaux $Z_\varepsilon^{(6,4,2)}$. In the case of SO(7) use may be made of figure 2 but the result has been given very recently by Gaskell (1983). Since each term of (5.26) contains a single numerator factor, rather than a sum of such factors, the application of (5.26) to SO(7) compare favourably with the result given explicitly by Gaskell. However, the method of derivation used by Gaskell is not based on Young tableaux and may be applied to each semi-simple Lie group. It is interesting to note that the relevant recurrence technique is reminiscent of that used in building the Young tableaux in the first instance (King and El-Sharkaway 1983) so that the two methods may not be as unrelated as might seem to be the case at first sight.

As a final comment it may be pointed out that each character generator can be re-expressed in the form:

$$X_C(\boldsymbol{\alpha}, \boldsymbol{\phi}) = N_G(\boldsymbol{\alpha}, \boldsymbol{\phi})/D_G(\boldsymbol{\alpha}, \boldsymbol{\phi}) \tag{6.5}$$

with

$$D_G(\boldsymbol{\alpha}, \boldsymbol{\phi}) = \prod_{\boldsymbol{x} \in \boldsymbol{P}(k)} [1 - l(\boldsymbol{x})]. \tag{6.6}$$

Multiplying numerator and denominator by additional factors $[1 + l(\boldsymbol{x})]$ for each $\boldsymbol{x}$ such that $l(\boldsymbol{x})$ involves $s = \frac{1}{2}$, enables the denominator $D_G(\boldsymbol{\alpha}, \boldsymbol{\phi})$ given by (6.6) to be evaluated on the assumption that $s = 1$. It then follows from the enumeration of poset demands described in § 3 that

$$D_G(\boldsymbol{\alpha}, \boldsymbol{\phi}) = \prod_{\lambda_p} \left\{ \sum_{q=1}^{d(\lambda_p)} (-1)^q A_p^q(\sigma) \chi^{\lambda_p \otimes \{1^q\}}(\boldsymbol{\phi}) \right\}, \tag{6.7}$$

where the product is taken over the elementary representations λ_p listed in (3.1), $d(\lambda_p)$ is the dimension of λ_p and $\lambda_p \otimes \{1^q\}$ is the qth rank antisymmetrised product of λ_p. In deriving this, use has been made of the fact that

$$\sum_{i=1}^{n} (1 - At_i) = \sum_{q=1}^{n} (-1)^q A^q e_q(t_1, t_2 \ldots t_n) \tag{6.8}$$

where e_q is the elementary symmetric function of degree q (Littlewood 1950, Macdonald 1979). Thus

$$\chi^{\lambda_p \otimes \{1^q\}}(\boldsymbol{\phi}) = e_q(t_1(\boldsymbol{\phi}), \ldots, t_p(\boldsymbol{\phi})) \tag{6.9}$$

where $t_1(\boldsymbol{\phi}), \ldots, t_p(\boldsymbol{\phi})$ is the sequence, with repetitions, obtained from (5.25) by considering all the entries $\eta(i, 1)$ in those poset elements consisting of a column of length p.

This form (6.7) of the denominator is that used to great effect by Moody *et al* (1982). Unfortunately it does not seem possible to obtain the numerator $N_G(\boldsymbol{\alpha}, \boldsymbol{\phi})$ so readily.

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